Dirac Delta Function

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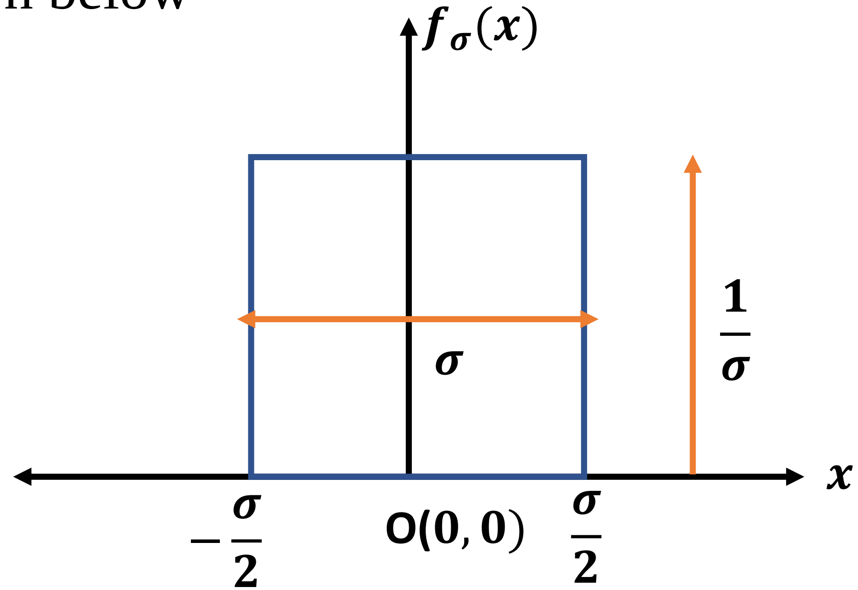
What is Dirac Delta Function ?

Let's assume a rectangular (or box) function

$$f_\sigma(x) = \begin{cases} \dfrac{1}{\sigma}, & -\dfrac{\sigma}{2} \leq x \leq \dfrac{\sigma}{2} \\ 0, & Otherwise \end{cases}$$

The plot of the function is shown below



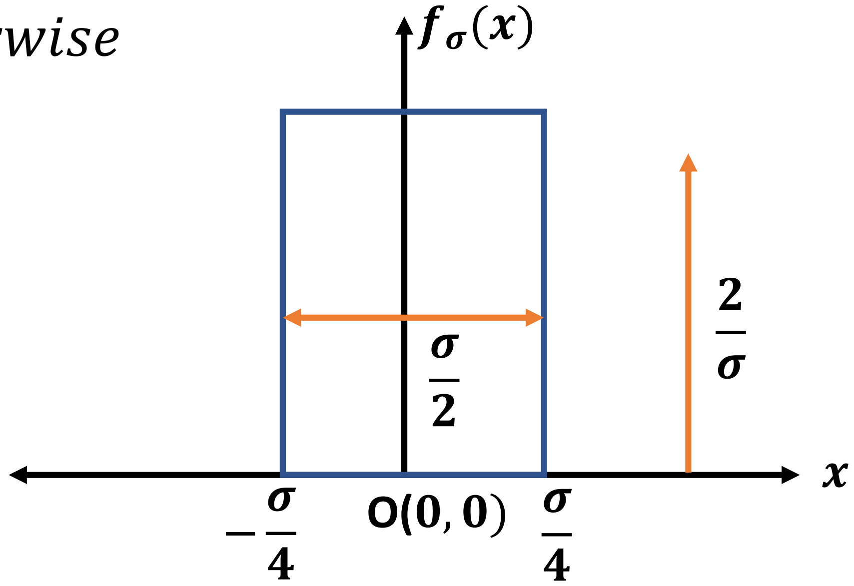
The area under the curve is equal to 1 (Area= width x height = $\sigma \times \frac{1}{\sigma} = 1$)

What is Dirac Delta Function ?

If the width is decreased to $\frac{\sigma}{2}$

$$f_\sigma(x) = \begin{cases} \frac{2}{\sigma}, & -\frac{\sigma}{4} \le x \le \frac{\sigma}{4} \\ 0, & Otherwise \end{cases}$$

The plot of the function is shown below



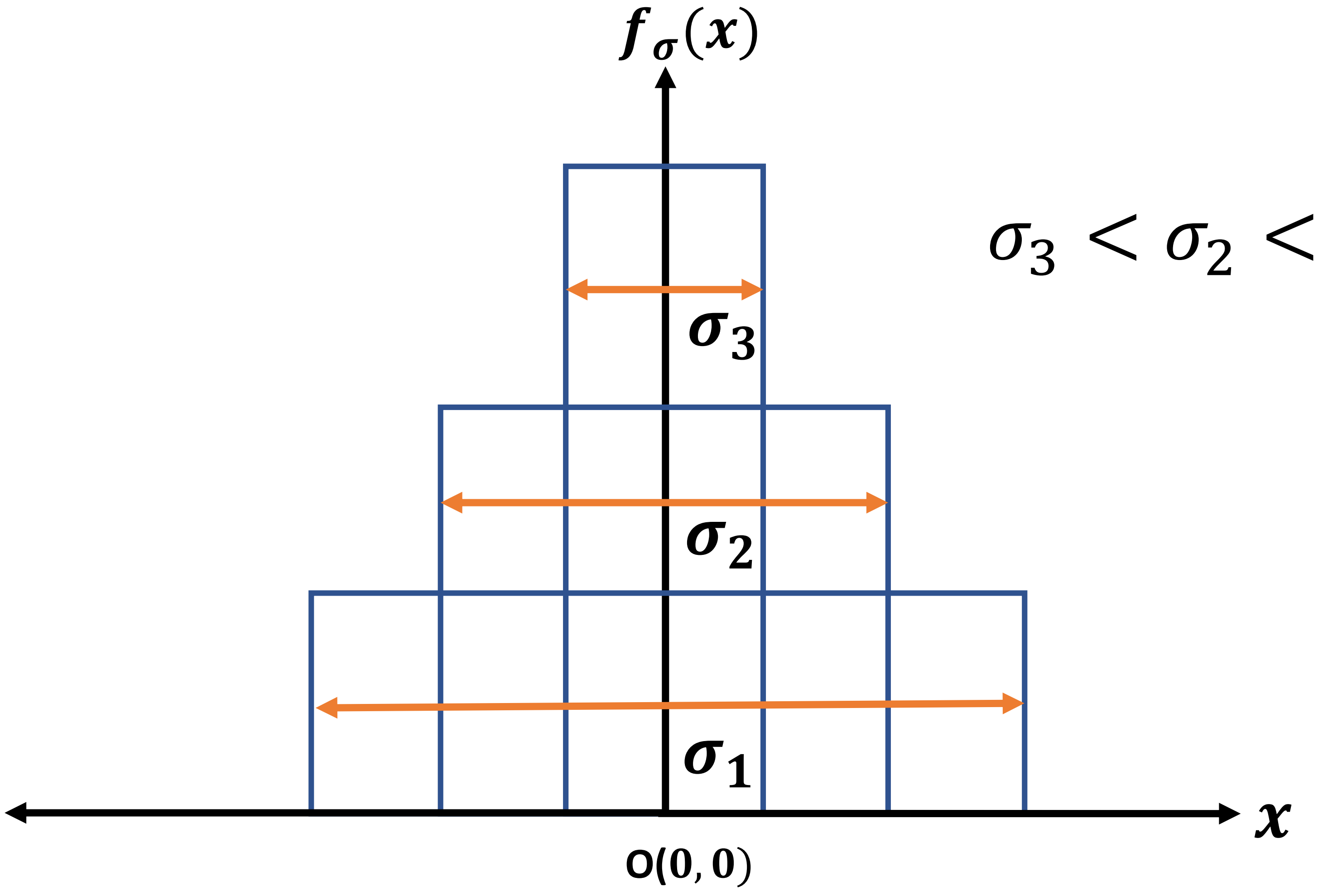
The area under the curve is still equal to 1

(Area= width x height = $\frac{\sigma}{2} \times \frac{2}{\sigma} = 1$)

If σ decreases, the function becomes narrower and taller. At the limit $\sigma \to 0$ the function becomes a Dirac delta function.

What is Dirac Delta Function ?

The plot of the rectangular function for different values of σ



$$\sigma_3 < \sigma_2 < \sigma_1$$

What is Dirac Delta Function ?

The value of the integration of the function in the region $-\frac{\sigma}{2} \leq x \leq \frac{\sigma}{2}$ is 1. Since the product of width and height of the rectangular function is one. We can write

$$\int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} f_\sigma(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty} f_\sigma(x)dx = 1$$

Since the function is zero outside the region $-\frac{\sigma}{2} \leq x \leq \frac{\sigma}{2}$.

As $\sigma \to 0, f_\sigma(x) \to \infty$, that is if the width of the rectangular function goes to zero, its height goes to infinity. The function becomes a tall narrow spike (impulse) which is called the Dirac delta function.

We can write

$$\delta(x) = \lim_{\sigma \to 0} f_\sigma(x)$$

Definition of Dirac Delta Function

So the Dirac delta function can be defined as

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

And

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

The Dirac delta function is defined at x = 0. For any other point x_0 Dirac delta function can be defined as

$$\delta(x - x_0) = \begin{cases} \infty, & x = x_0 \\ 0, & x \neq x_0 \end{cases}$$

And

$$\int_{-\infty}^{\infty} \delta(x - x_0)dx = 1$$

Properties of Dirac Delta Function

1.

$$\int_{-\infty}^{\infty} \boldsymbol{f(x)\delta(x-x_0)dx = f(\mathbf{x_0})}$$

where f(x) is any arbitrary function.

Proof: Let's assume any function f(x) and the rectangular function $f_\sigma(x)$.

Now

$$\int_{-\infty}^{\infty} f(x)f_\sigma(x)dx = \int_{x_0-\frac{\sigma}{2}}^{x_0+\frac{\sigma}{2}} f(x).\frac{1}{\sigma}.dx = \frac{1}{\sigma}\int_{x_0-\frac{\sigma}{2}}^{x_0+\frac{\sigma}{2}} f(x).dx$$

At $\sigma \rightarrow 0$, the integral

$$\frac{1}{\sigma}\int_{x_0-\frac{\sigma}{2}}^{x_0+\frac{\sigma}{2}} f(x).dx \approx \frac{1}{\sigma}.\sigma f(x_0) = f(x_0)$$

This is called shifting property of Dirac delta function which tells that the product of the Dirac delta function with any function is to pick the value of the function at the point where the Dirac delta function is defined.

Properties of Dirac Delta Function

2.

$$\boldsymbol{\delta(ax) = \frac{1}{|a|}\delta(x), \qquad a \neq 0}$$

Proof:

Let's assume the integral $\int_{-\infty}^{\infty} f(x)\delta(ax)dx$. Let $y = ax$, then $dy = adx \Rightarrow dx = dy/a$

$$\int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y).\frac{dy}{a} = \frac{1}{a} \times \int_{-\infty}^{\infty} f\left(\frac{y}{a}\right)\delta(y)dy = \frac{1}{a}f(0) = \frac{1}{a}\int_{-\infty}^{\infty} f(x)\delta(x)dx$$

3.

$$\boldsymbol{\delta(-x) = \delta(x)}$$

Proof:

Let's assume the integral $\int_{-\infty}^{\infty} f(x)\delta(-x)dx$. Let $y = -x$, then $dy = -dx \Rightarrow dx = -dy$

$$\int_{\infty}^{-\infty} f(-y).\delta(y).-dy = \int_{-\infty}^{\infty} f(-y)\delta(y)dy = f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx$$

Properties of Dirac Delta Function

4.

$$\boldsymbol{\delta(a-x)=\delta(x-a)}$$

Proof:

Let's assume the integral $\int_{-\infty}^{\infty} f(x)\delta(a-x)dx$. Let $y=a-x$, then $dy=-dx \Rightarrow dx=-dy$

$$\int_{-\infty}^{\infty} f(x)\delta(a-x).dx \ = f(a) = \int_{\infty}^{-\infty} f(a-y).\delta(y).-dy = \int_{-\infty}^{\infty} f(a-y)\delta(y)dy = f(a) = \int_{-\infty}^{\infty} f(x)\delta(x-a)dx$$

5.

$$\boldsymbol{\int_{-\infty}^{\infty} \delta(x-a)\,\delta(x-b)dx = \delta(a-b)}$$

Proof:

Let's assume $x-b=y$ then $dx=dy$

$$\int_{-\infty}^{\infty} \delta(y+b-a)\delta(y).dx \ = \delta(b-a) = \delta(a-b)$$

Properties of Dirac Delta Function

6.

$$\boldsymbol{x\delta'(x) = -\delta(x)}$$

Proof:

Let's assume the integral $\int_{-\infty}^{\infty} f(x).x\frac{d}{dx}(\delta(x)dx$

$$\int_{-\infty}^{\infty} f(x).x\frac{d}{dx}\delta(x)dx = [xf(x)\delta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}(xf(x)\delta(x)dx$$

$$\Rightarrow 0 - \int_{-\infty}^{\infty} f(x)\delta(x)dx - \int_{-\infty}^{\infty} xf'(x)\delta(x)dx$$

$$\Rightarrow 0 - \int_{-\infty}^{\infty} f(x)\delta(x)dx - 0 = \int_{-\infty}^{\infty} f(x)\delta(x)dx$$

Properties of Dirac Delta Function

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$$\int_{-\infty}^{\infty} f(x).\,x\frac{d}{dx}\delta(x)dx = [xf(x)\delta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}(xf(x)\delta(x)dx$$

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$$\Rightarrow 0 - \int_{-\infty}^{\infty} f(x)\delta(x)dx - 0 = \int_{-\infty}^{\infty} f(x)\delta(x)dx$$

Properties of Dirac Delta Function

7.

$$\boldsymbol{\delta(f(x) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}}$$

Proof:

We know $\boldsymbol{\delta(ax) = \frac{1}{|a|}\delta(x)}$ if

$$\int_{-\infty}^{\infty} \delta(ax)f(x)dx = \frac{1}{|a|}f(0) = \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)\delta(x)dx$$

The value of the function f(x) is evaluated at x = 0. So for $f(x)\delta(x - x_0)$ the function is evaluated at $x = x_0$. x_0 is the root of the function f(x). If f(x) has n number of roots and x_i is the i^{th} root then we can write

$$\delta(kx) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|a|}$$

So that

$$\delta(f(x) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}$$

Properties of Dirac Delta Function

8.

$$\boldsymbol{\delta(x^2 - a^2) = \frac{\delta(x-a)}{|2a|} + \frac{\delta(x+a)}{|2a|}}$$

Proof:

We know

$$\delta(f(x) = \sum_{i=1}^{n} \frac{\delta(x - x_i)}{|f'(x_i)|}$$

Here $f(x) = x^2 - a^2 = (\mathrm{x} - \mathrm{a})(\mathrm{x} + \mathrm{a})$ and $f'(x) = 2x$.

Roots of the function are $\pm a$.

So that

$$\delta(x^2 - a^2) = \frac{\delta(x-a)}{|f'(a)|} + \frac{\delta(x+a)}{|f'(-a)|} = \frac{\delta(x-a)}{|2a|} + \frac{\delta(x+a)}{|2a|}$$